# Synopsis

Modern engineering abounds in problems characterised by axisymmetry  Contact problems between elastic bodies are also of immense engineering interest  In these problems, the effect of a load is significant only in the contact vicinity and may cause cracking of the material in the region of influence  To compound the problem further, the invasion of composites into load bearing and power transmitting components demands a thorough examination of singularities at corners and joints  In fibre-reinforced composites, fibre pull-out and push-in tests are performed to determine the interface characteristics  In the push-in test, a single fibre embedded in a matrix is subjected to an indentation load and any slip at the fibre-matrix interface is detected  Though this test was proposed nearly a decade ago, a comprehensive analysis of the stress field is not available  Various approximations do exist, however, a rigorous elasticity solution is desirable  A similar need also applies in assessing the mechanical integrity of optical fibres which are compound cylinders  In civil engineering, the problem of an elastic half-space weakened by a cylindrical hole with axis normal to the plane surface has wide-ranging applications  The problem of a lined hole is also of significance  Earlier researchers have approximated the lining as rigid or as a shell  Such approximations may not capture reality, and therefore a rigorous axisymmetric analysis will be a valuable contribution

Complementing the problems of *torsionless* axisymmetry cited above and much simpler than them in analysis are the corresponding *torsional* problems  Torsional axisymmetric loading arises frequently in contact problems when there is relative rotation accompanied with friction  These two classes of problems – torsionless and torsional axisymmetry – can be decoupled  It is instructive to tackle torsional problems before attempting more complex torsionless problems in axisymmetry

Problems listed above with wide-ranging applications have one salient unifying feature – they are all *axisymmetric*  A unified method based on integral transforms can tackle the above problems  The mathematical tools which find use in these problems are Fourier sine / cosine, Hankel and extended Hankel transforms A review of literature on integral transforms reveals that generating numerical results about stress and displacement fields is quite challenging particularly with respect to torsionless problems

Singularities that may exist around edges along bonded bimaterial interfaces are important to study possible sites for crack initiation These singularities pose consider able numerical problems in global studies and require local analysis Such a study for axisymmetric composite systems has not been attempted in the literature

On the experimental front, the method of scattered light photoelasticity offers a non-destructive tool for the analysis of axisymmetric problems Three dimemsional photoelasticity with slicing has been used to study fibre-matrix problems However, this method is destructive in nature It is desirable that a device be constructed for whole-field visualisation of scattered light fringes in axisymmetric specimens

It is the objective of this thesis to develop rigorous elasticity solutions of importance to axisymmetric contact mechanics problems The first is that of an elastic half space weakened by a circular cylindrical hole with axis normal to the flat surface of the half space This hole reinforced with an elastic lining constitutes the second problem On the other hand, this hole filled with a core of different elastic properties, like a fibre in an infinite matrix, forms the third problem in this study Now, when the matrix is a cylindrical sheath as in an optical fibre, the fourth problem is described Above problems are examined in the context of both torsional and torsionless loading Numerical results are provided for all four torsional problems and two representative torsionless problems, and formulations are given for the remaining two problems in torsionless axisymmetry Possible stress functions for the problem of load acting in a cylindrical depression in a half-space, as in an embedded foundation, are also indicated At the junction of two materials forming a corner, singularities are encountered An axisymmetric examination of such singularities to understand the complete stress field is also a significant part of this thesis Lastly, scattered light fringe patterns from a finite cylinder under ball indentation provide an experimental flavour to this thesis This thesis is organised as follows

Chapter One provides the motivation to this thesis and outlines the aim and scope of this work

Chapter Two contains a review of literature on axisymmetric elasticity A brief history of half-space problems is outlined first Then literature on semi-infinite solid cylinders, half-space with cylindrical hole, elastic fibre embedded in an elastic matrix, compound cylinders, singularities in axisymmetry and experimental approaches to axisymmetric problems is discussed

Chapter Three presents the basic theory of axisymmetric elasticity Also presented are integral transform methods for tackling boundary value problems Functional methods for handling axisymmetric problems, especially Love stress function, are highlighted A catalogue of useful stress and displacement functions to tackle both torsional and torsionless problems is then provided This is followed by a brief discussion on numerical methods for solution of integral equations

The above three chapters provide the motivation, rationale and necessary analytical tools to tackle unsolved problems attempted in the remainder of this thesis The main thrust of this thesis is directed toward solution of traction boundary value problems as opposed to mixed boundary value problems commonly attempted However, formulations provided here are general enough to accommodate mixed boundary conditions

Chapter Four begins with the torsional problem of an elastic half-space containing a cylindrical hole Boundary conditions are applied by means of known tractions One of the main intentions of this chapter is to demonstrate the use of Hankel transforms for solving axisymmetric problems Two different formulations using the Hankel transform and the extended Hankel transform are shown to lead to identical results for stress and displacement Since the Hankel transform approach is numerically advantageous, subsequent formulations exclusively use the Hankel transform This chapter concludes with the torsion problem of a lined cylindrical hole in a half space for which typical results for stress and displacement fields are presented

Chapter Five continues with the half space torsion problem of an embedded elastic fibre For the special case when the fibre and matrix are made of the same material, the results compare well with the solution for a homogeneous half-space, derived separately Further, far from the region of loading, the results approach the Foppl solution for a concentrated torsional moment on a homogeneous half space Numerical results for dissimilar materials show a shear stress $\sigma_{\theta z}$ discontinuity at the interface which is explained This chapter concludes with the problem of a compound cylinder subjected to a torque at the end Far away from the loaded end, shear stresses in this problem approach the strength of-materials prediction elucidating Saint Venant's principle

Preceding chapters dealing with torsional axisymmetry provide the necessary background for tackling the more complex problems of torsionless axisymmetry considered in the following chapters Torsionless axisymmetric problems form the main thrust of contact mechanics Closed form results are seldom possible in this class of problems, numerical techniques for solving integral equations are a vital part of the solution as highlighted in the subsequent two chapters

Chapter Six revisits the problem of a half-space weakened by a circular cylindrical hole in the torsionless case Using Love stress functions, the governing integral equation is derived and solved numerically guided by the techniques illustrated earlier The integral formulation of the problem of a lined hole concludes this chapter

Chapter Seven analyses the push-in test when the cylindrical cavity in Chapter Six is filled with an elastic fibre having different elastic properties The push-in load is an applied normal traction on the flat surface of the core The load transfer behaviour is studied assuming the interface as perfect A set of coupled integral equations is deduced and solved numerically Continuity of tractions and displacements across the interface is verified Upon specialising the dissimilar half-space to a homogeneous half-space, the well-known Hertz solution is recovered The integral formulation to model load transfer in optical fibres ends this chapter

A critical aspect in composite materials concerns stress singularities at corners forming the junction as mentioned earlier The well known William's approach extended by Zak to axisymmetric problems is developed as a part of this investigation in Chapter Eight This is essential for a complete understanding of the stress field for composite problems addressed in this thesis The dominant singularity in geometries and boundary conditions considered here coincides with the corresponding plane strain results

Chapter Nine outlines an experimental investigation using the technique of scattered light photoelasticity to visualize the unique features of axisymmetric stress fields The *three incidence method* for determining stresses using scattered light photoelasticity is briefly discussed A compact scattered light polariscope is fabricated to automate whole field data acquisition using a laser beam traverse mechanism Scattered light fringe patterns obtained for an axisymmetric specimen are displayed to demonstrate the capability of the facility

Chapter Ten summarises all the major findings of this thesis Suggestions for further work are also enlisted

Appendix A contains pertinent mathematical results about Bessel functions and integral transforms Also included are useful antiderivatives and definite integrals used at various stages of this thesis Stress functions to embedded foundation problems are presented in Appendix B Appendix C includes some intermediate steps omitted in Chapters Six and Seven concerning torsionless axisymmetric problems

Thus, this thesis presents some theoretical advances to tackle problems in axisymmetric elasticity of relevance to contact mechanics These problems are studied under two main headings – torsional and torsionless Emphasis is given to composite systems Globally valid stress-field solutions obtained using integral transforms are complemented with an examination of local singularities at edges in the composites Lastly, the development of an experimental facility for scattered light photoelasticity to assist in axisymmetric stress analysis is also presented